4680 - HW 3

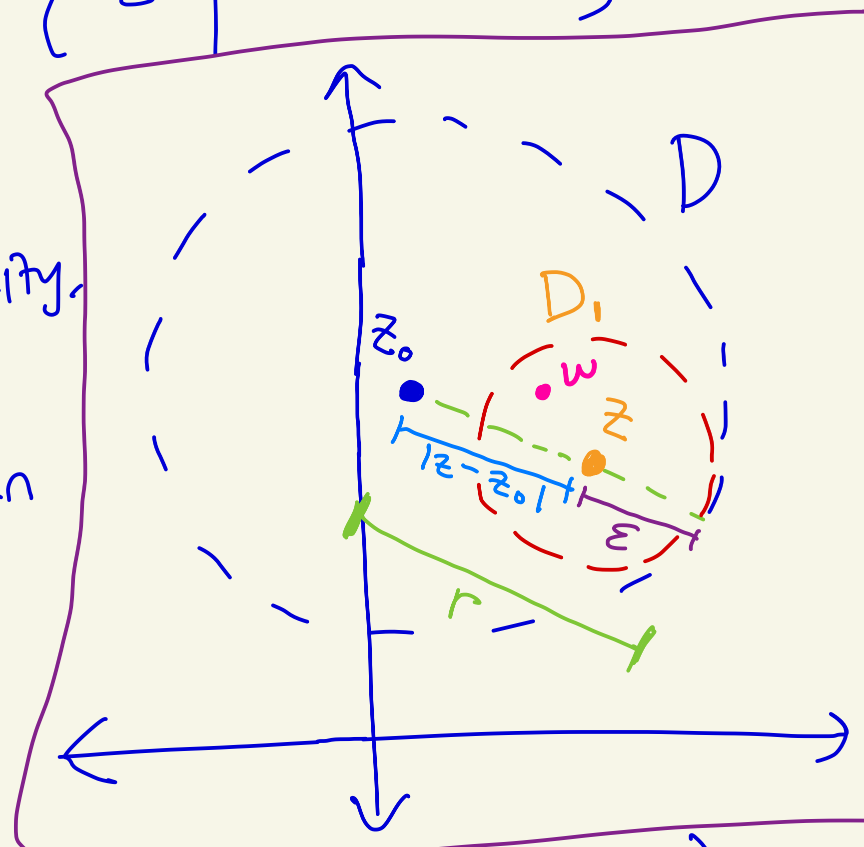
Solutions

$\boxed{1}$ We want to show that the r-neighborhood

$$D(z_0; r) = \{ z \mid |z - z_0| < r \}$$

is open.

Write $D = D(z_0; r)$ for notational simplicity.



Pick some $z \in D$.

We now show z is an interior point of D and since z was arbitrarily chosen, D is open.

Set $\varepsilon = r - |z - z_0|$. (See the picture.)

We will show that $D_1 = D(z; \varepsilon) \subseteq D$ which shows that z is an interior point. (Recall $D_1 = \{ w \mid |w - z| < \varepsilon \}$)

Let $w \in D_1$.

Then $|w - z_0| = |w - z + z - z_0| \leq$

$$\leq |w - z| + |z - z_0| \underset{w \in D_1}{<} \varepsilon + |z - z_0|$$

$$= r - |z - z_0| + |z - z_0| = r.$$

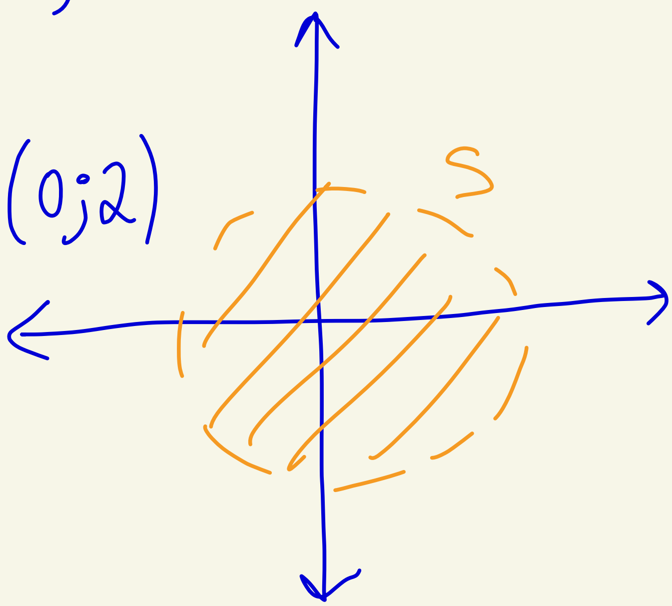
So, $|w - z_0| < r$ and hence $w \in D$. So, $D_1 \subseteq D$.

$\boxed{2(a)}$ S is open but not closed.

Why? By problem 1, the r-neighborhood

$S = \{ z \mid |z| < 2 \} = D(0;2)$

is open.



Is S closed? No.

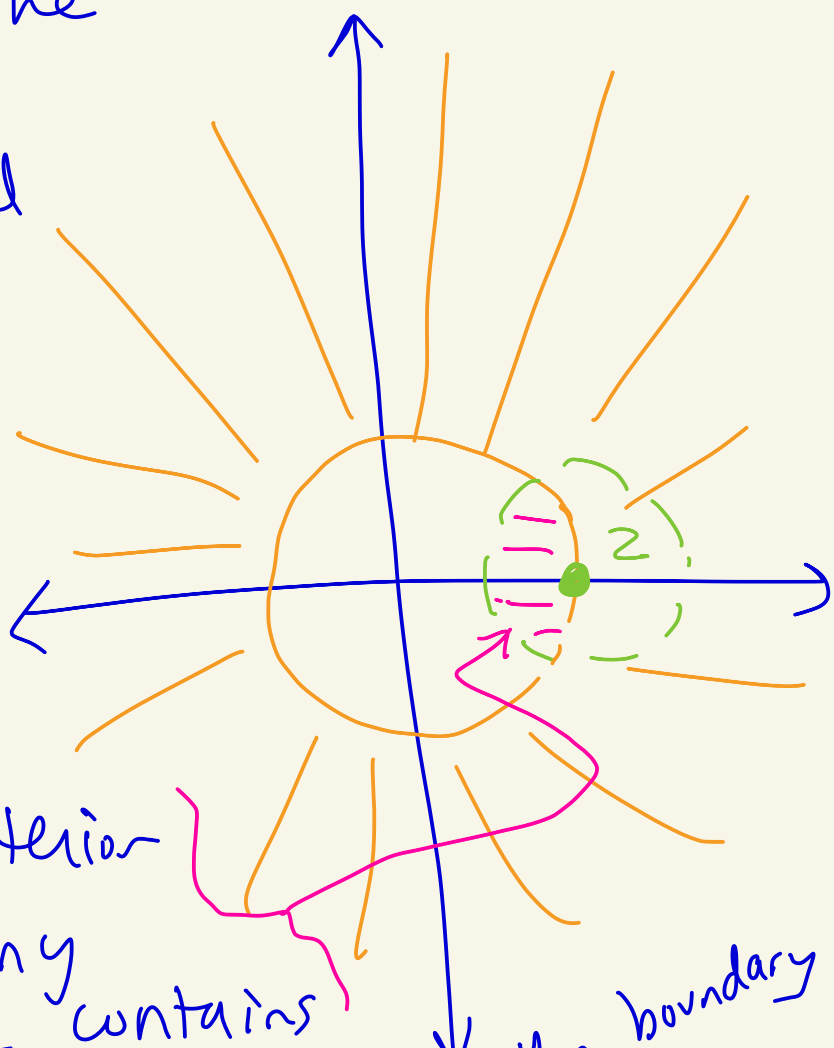
Let $T = \mathbb{C} - S$, the complement of S.

For S to be closed we would need T to be open.

But it isn't.

For example $2 \in T$ but

2 is not an interior point of T since any ε-neighborhood of 2 contains points outside of T. Ie, 2 is on the boundary of T.

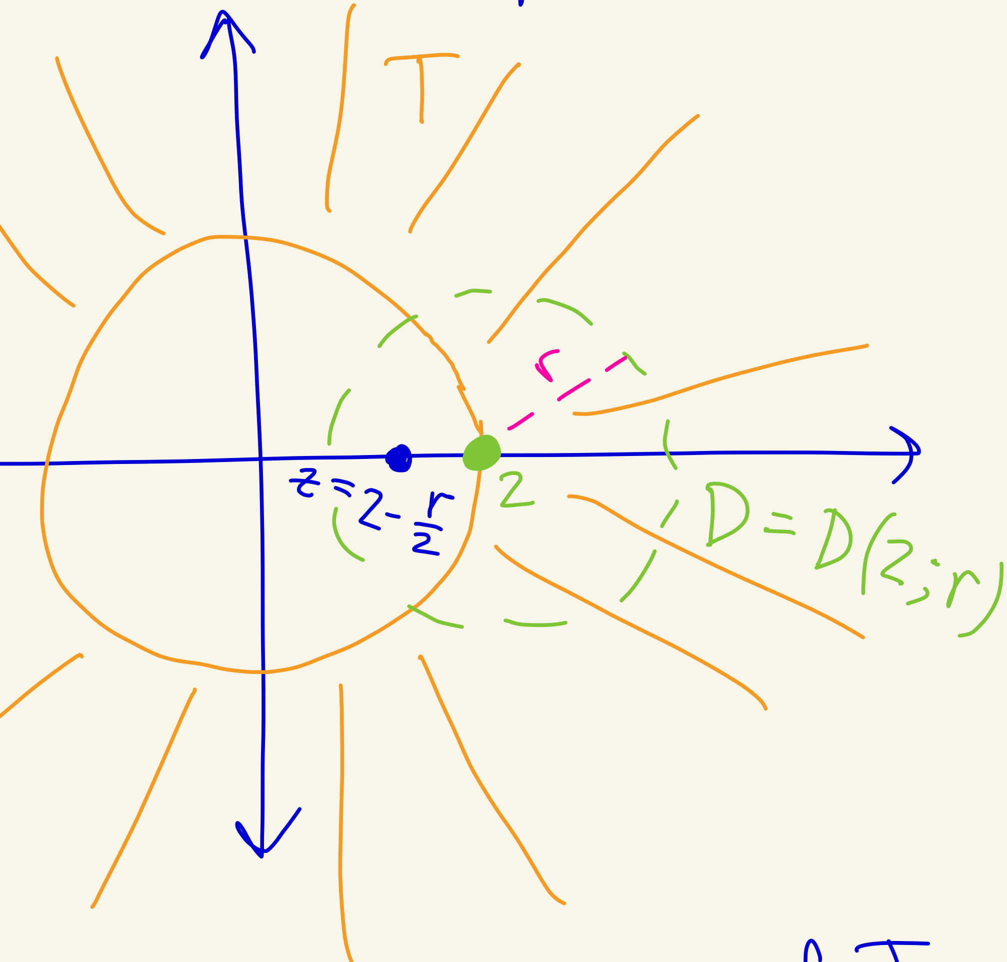


See the next page why this is true.

Let's see how we could prove this formally.

Let $r > 0$.

We show that $D = D(2;r)$ is not completely contained in T no matter what r is.



So, 2 is not an interior point of T and T is not open.

We may assume that $r < 1$ since shrinking ε just makes a smaller disc.

Let $z = 2 - \frac{r}{2}$.

Then $|z - 2| = |-\frac{r}{2}| = \frac{r}{2} < r$. So, $z \in D(2;r)$,

Note that $2 - \frac{r}{2}$ is a real number and $2 - \frac{r}{2} > 0$ [since $r < 1$]. So,

$|z| = |2 - \frac{r}{2}| = 2 - \frac{r}{2} < 2$. So, $z \notin T$.

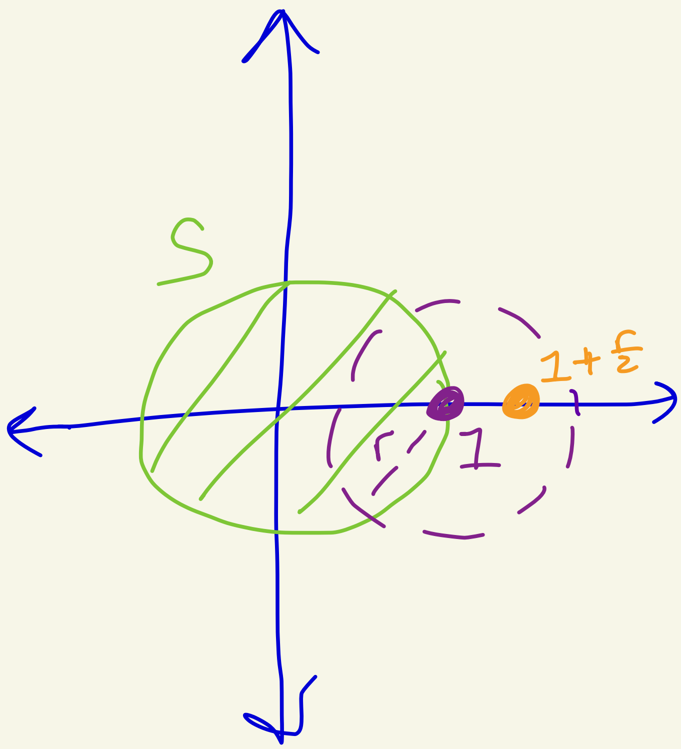
Since $z \in D(2;r)$ but not in T, and r was arbitrary, 2 is not an interior point of T.

$\boxed{2(b)}$ Let $S = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$.

Let's first show that S is not open.

Consider $1 \in S$.

Let $r > 0$ and consider $D(1; r)$.



Then $1 + \frac{r}{2} \in D(1; r)$ but $1 + \frac{r}{2} \notin S$.

So there is no disc $D(1; r)$ totally contained in S.

Thus, $1 \in S$ but 1 is not an interior point of S.

So, S is not open.

We show that S is closed.
We give two methods

method 1

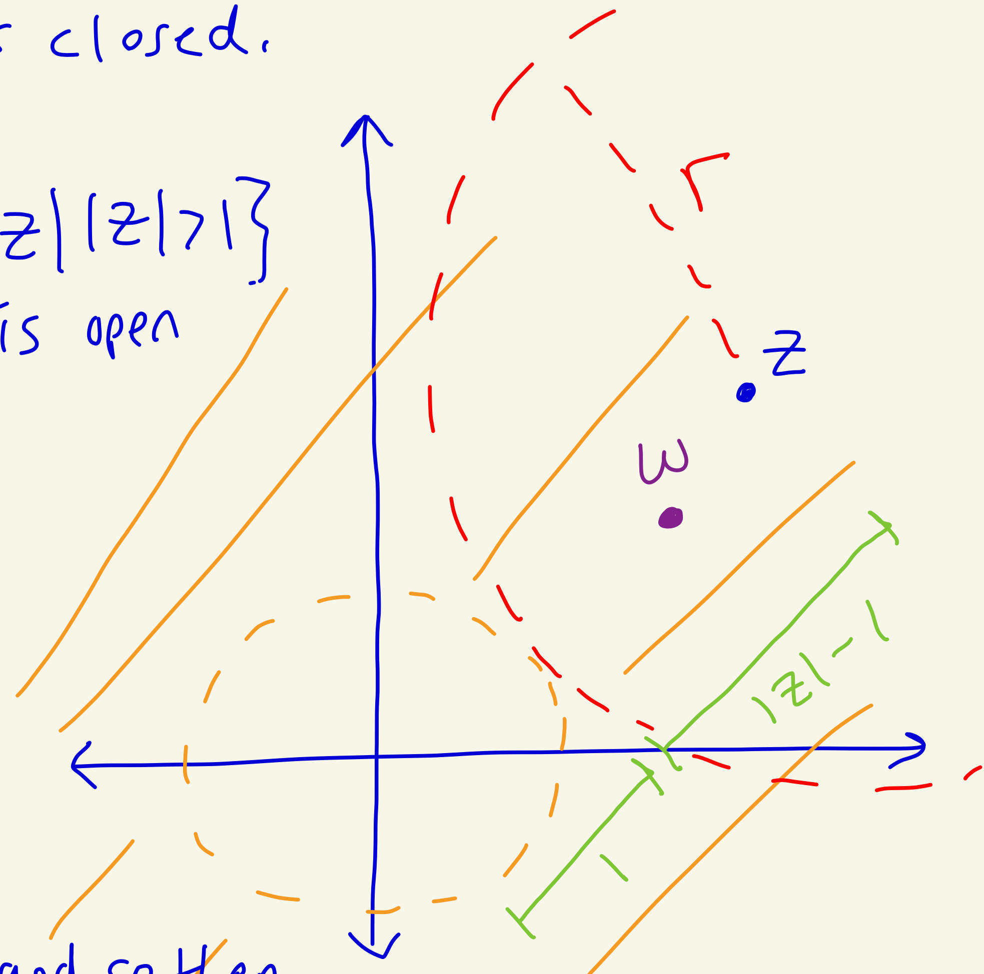
Let $T = \mathbb{C} - S = \{z \mid |z| > 1\}$

If we show that T is open then S is closed.

Pick some $z \in T$.

Let $r = |z| - 1$.

Consider the disc $D(z;r)$.



We will show that $D(z;r) \subseteq T$, and so then z is an interior point of T. Since z is arbitrary this shows that T is open.

Let $w \in D(z;r)$

We will show that $|w| > 1$ and hence $w \in T$.

Suppose instead that $|w| \leq 1$.

If that were the case then

$$|z| = |z - w + w| \leq |z - w| + |w|$$
$$< r + |w|$$
$$= |z| - 1 + |w|$$
$$\leq |z| - 1 + 1 \leq |z|.$$

since $w \in D(z;r)$

But then $|z| < |z|$. Contradiction. Hence $|w| > 1$ and $w \in T$. So T is open.

Method 2

Here's another way to show that $T = \mathbb{C} - S = \{ z \mid |z| > 1 \}$ is open and hence $S = \{ z \mid |z| \leq 1 \}$ is closed.

(Use the same picture as the previous page)

Let $z \in T$

Let $r = |z| - 1$.

Since $|z| > 1$ we have that $r > 0$.

We will show that $D(z;r) \subseteq T$ and hence z is an interior point of T. This will imply that T is open since z was arbitrary.

Let $w \in D(z;r)$.

Then $|z - w| < r$.

Thus,
$$|z| = |z - w + w| \leq |z-w| + |w| < r + |w| = |z| - 1 + |w|$$

So, $|z| < |z| - 1 + |w|$.

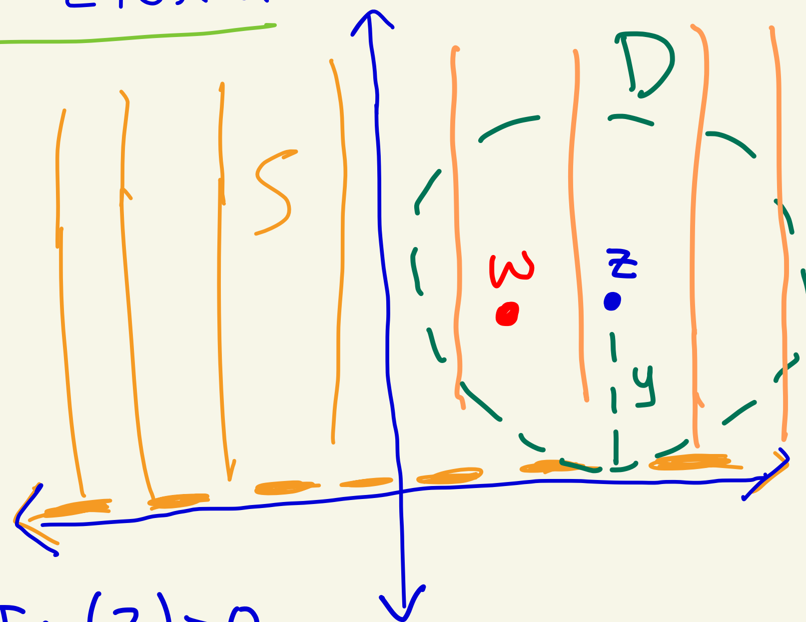
Thus, $1 < |w|$.

So, $w \in T$.

Thus, $D(z;r) \subseteq T$ as we wanted. $\square$

$\boxed{2(c)}$ $S = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

S is open. S is not closed.



Let's show S is open.

Let $z = x + iy \in S$.

We must show that z is an interior point of S.

Since $z \in S$ we know $y = \mathrm{Im}(z) > 0$.

Let $D = D(z; y)$

If we show that $D \subseteq S$ then this shows that z is an interior point of S.

$D = D(z; y) = \{ w \mid |w - z| < y \}$

Suppose $w \in D$.
We must show that $w \in S$.

Then $|w - z| < y$.

Suppose $w = x' + iy'$.

Plugging $w = x' + iy'$ and $z = x + iy$ into $|w - z| < y$ gives

$$\sqrt{(x' - x)^2 + (y' - y)^2} < y$$

Thus,

$$(x' - x)^2 + (y' - y)^2 < y^2$$

Thus, $(y'-y)^2 < y^2 - (x'-x)^2$

But $(x'-x)^2 \geq 0$, thus $y^2 - (x'-x)^2 \leq y^2$.

Thus, $(y'-y)^2 < y^2 - (x'-x)^2 \leq y^2$.

So, $(y'-y)^2 < y^2$.

Thus, $\sqrt{(y'-y)^2} < \sqrt{y^2} = y$

since $y > 0$

$\sqrt{t^2} = |t|$

So, $|y'-y| < y$

Thus, $-y < y'-y < y$

recall: If $a,b,c \in \mathbb{R}$ and $c > 0$, then $|a-b| < c$ iff $-c < a-b < c$

So, $0 < y' < 2y$.

Thus, $0 < y' = \text{Im}(w)$.

So, $w \in S$.

Thus, $D \subseteq S$.

So, z is an interior point of S.

So, S is open.

Note: You do part of this proof in another way.

Use:

$$\begin{aligned} |y'-y| &= |\text{Im}(w) - \text{Im}(z)| \\ &= |\text{Im}(w-z)| \\ &\leq |w-z| \\ &< y \end{aligned}$$

Thus, $|y'-y| < y$.

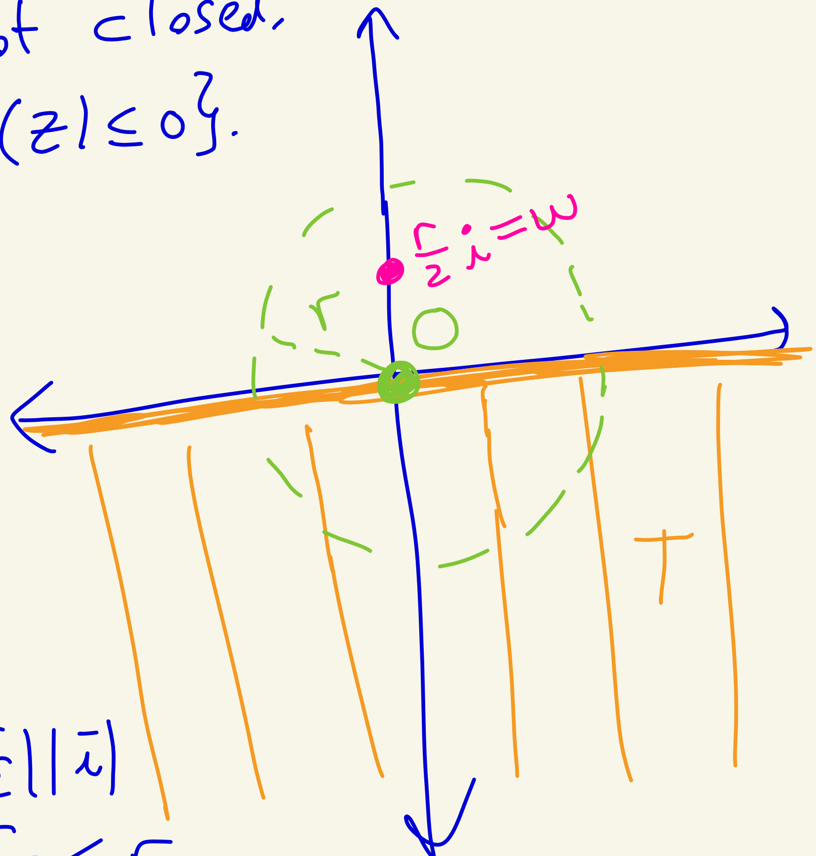
Then proceed as before

We show that S is not closed.

Let $T = \mathbb{C} - S = \{z \mid \mathrm{Im}(z) \leq 0\}$.

Then, $0 \in T$.

We show that 0 is not an interior point of T and hence T is not open.



Let $r > 0$.

Let's show $D(0;r) \nsubseteq T$.

Consider $w = \frac{r}{2}i$.

Then, $|w - 0| = |\frac{r}{2}i| = |\frac{r}{2}||i|$
$= \frac{r}{2} < r$.

Thus, $w \in D(0;r)$.

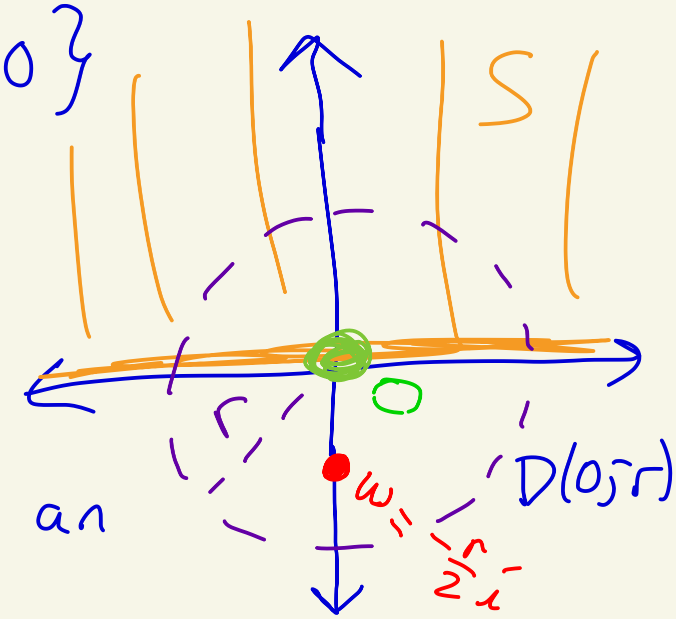
However, $\mathrm{Im}(w) = \mathrm{Im}(0 + \frac{r}{2}i) = \frac{r}{2} > 0$.

So, $w \notin T$.

Thus, $D(0;r) \nsubseteq T$.

$\boxed{2(d)}$

$S = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0 \}$



Let's show that S is not open.

Note that $0 \in S$.

We will show that 0 is not an interior point of S and so S is not open.

We will show that no disc centered at 0 is completely contained in S.

Let $r > 0$. Consider $D(0;r)$.

Let $w = -\frac{r}{2}i$.

Note that $\underbrace{|w-0|}_{\text{distance between } w \text{ and } 0} = \left|-\frac{r}{2}i\right| = \left|-\frac{r}{2}\right|\underbrace{|i|}_{1} = \frac{r}{2} < r$

So, $w \in D(0;r)$

And, $\mathrm{Im}(w) = \mathrm{Im}\left(0 - \frac{r}{2}i\right) = -\frac{r}{2} < 0$.

So, $w \notin S$.

Thus, no matter what $r > 0$ we pick $D(0;r) \not\subseteq S$.

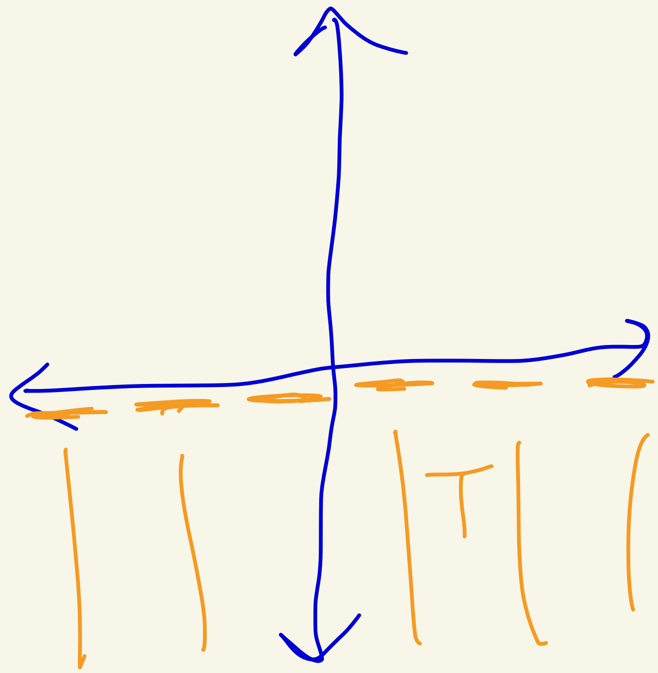
So, $0 \in S$ but not an interior point of S.

So, S is not open.

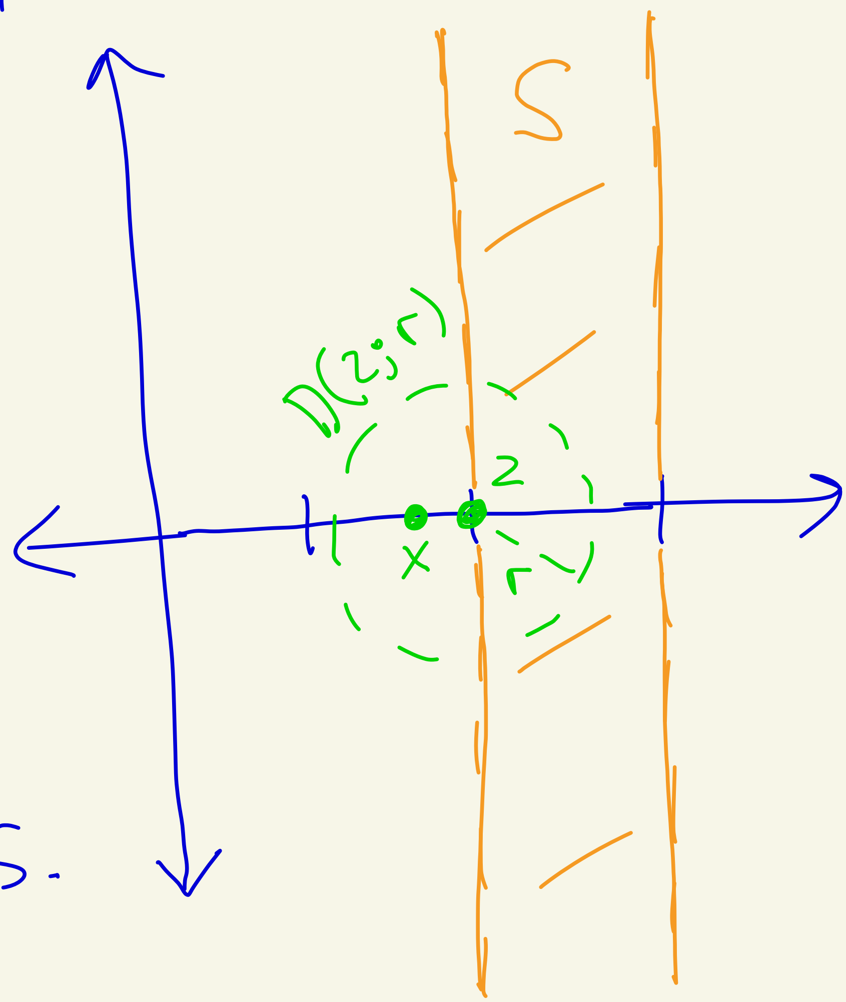
S is closed.

$T = \mathbb{C} - S$ is open

The proof is the same as how we showed the S in 2(c) is open, except you'll need to look at $-y$ instead of y in your picture.



$\boxed{2(e)}$ $S = \{ z \in \mathbb{C} \mid 2 \leq \mathrm{Re}(z) \leq 3 \}$



Let's show that S is not open.

Consider $2 \in S$.

We will show that 2 is not an interior point of S. And this shows S is not open.

Let $r > 0$.

We show $D(2;r) \not\subseteq S$.

Let $x = 2 - \frac{r}{4}$.

Then, $|x - 2| = \left|2 - \frac{r}{4} - 2\right| = \left|-\frac{r}{4}\right| = \frac{r}{4} < r$

So, $x \in D(2;r)$.

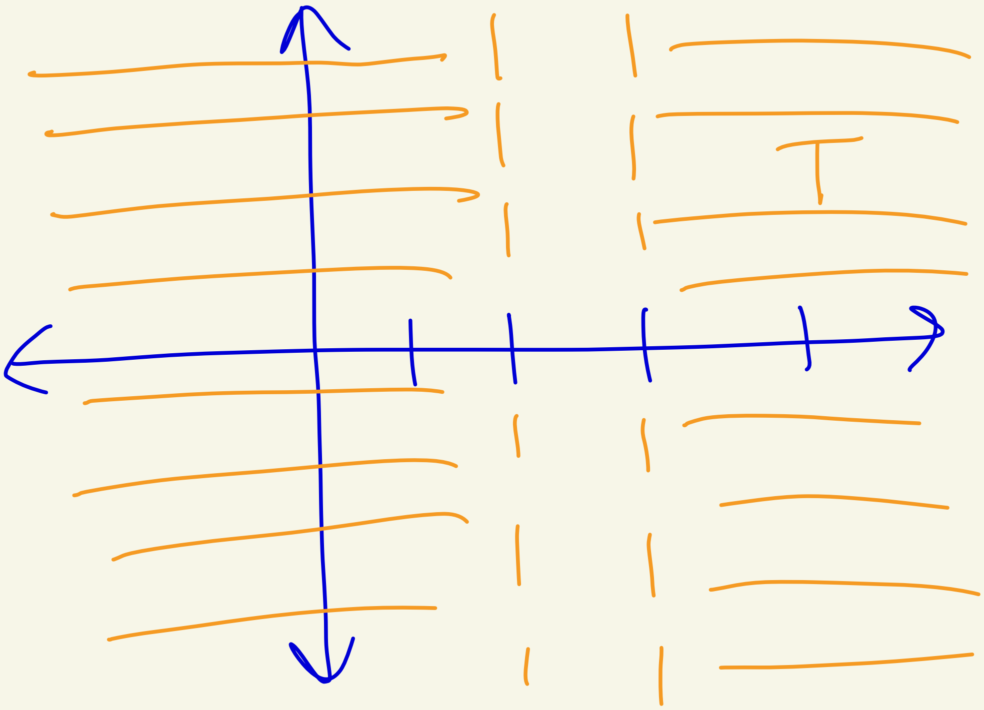
But $\mathrm{Re}(x) = \mathrm{Re}\left(2 - \frac{r}{4} + i0\right) = 2 - \frac{r}{4} < 2$

So, $x \notin S$.

Thus, $D(2;r) \not\subseteq S$.

You could also use $x = 2 - \frac{r}{2}$ or $x = 2 - \frac{3}{2}r$ or other choices

S is closed since $T = \mathbb{C} - S$ is open.

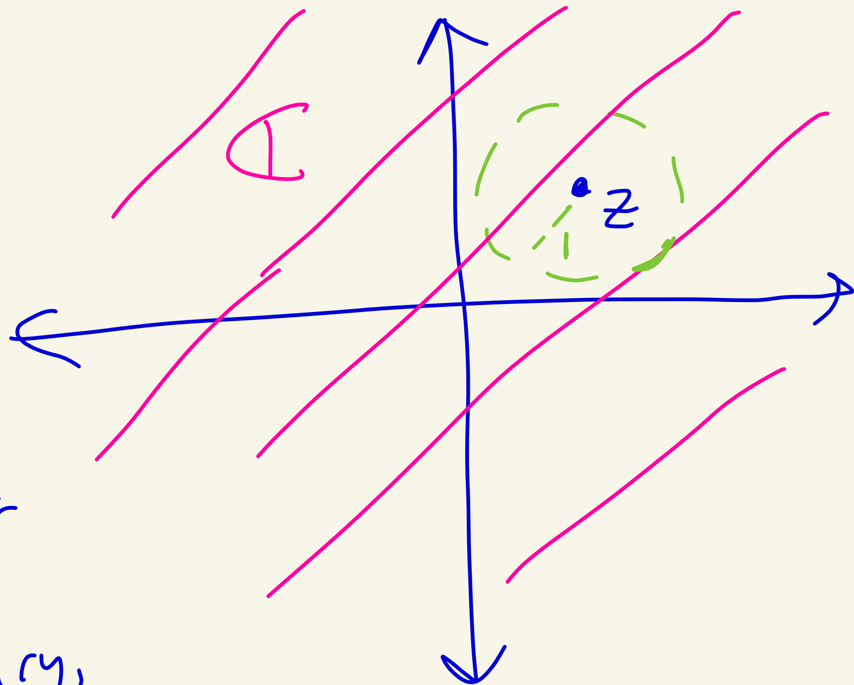


This proof would be in two parts. Show the left side is open and then show the right side is open [like in in 2(c) but more complicated]

Then the union of the two open sets is open and is T.

$\boxed{3(a)}$ $\mathbb{C}$ is open.

Why?



Let $z \in \mathbb{C}$.

Then $D(z;1) \subseteq \mathbb{C}$.

So, z is an interior point of $\mathbb{C}$.

Since z was arbitrary, $\mathbb{C}$ is open.

(You don't need to pick 1 as your radius, you could pick any $r > 0$).

3(b)

In logic a statement

$$(\forall x \in S)(P(x))$$

is true when $P(x)$ is true for every $x \in S$.

Think about the def of open:

$S \subseteq \mathbb{C}$ is open if the following is true:

$$(\forall z \in \mathbb{C})(\text{If } z \in S, \text{ then } z \text{ is an interior point of } S)$$

What if $S = \phi$? We have this statement:

$$(\forall z \in \mathbb{C})\underbrace{\left(\text{If } \underbrace{z \in \phi}_{\text{always false}}, \text{ then } z \text{ is an interior point of } S\right)}_{\text{If } P, \text{ then } Q \text{ always true since } P \text{ is always false}}$$

The overall statement is true, so ϕ is open.

$\boxed{3(c)}$ $\mathbb{C}$ is closed because
$\mathbb{C} - \mathbb{C} = \phi$ is open (by 3(b))

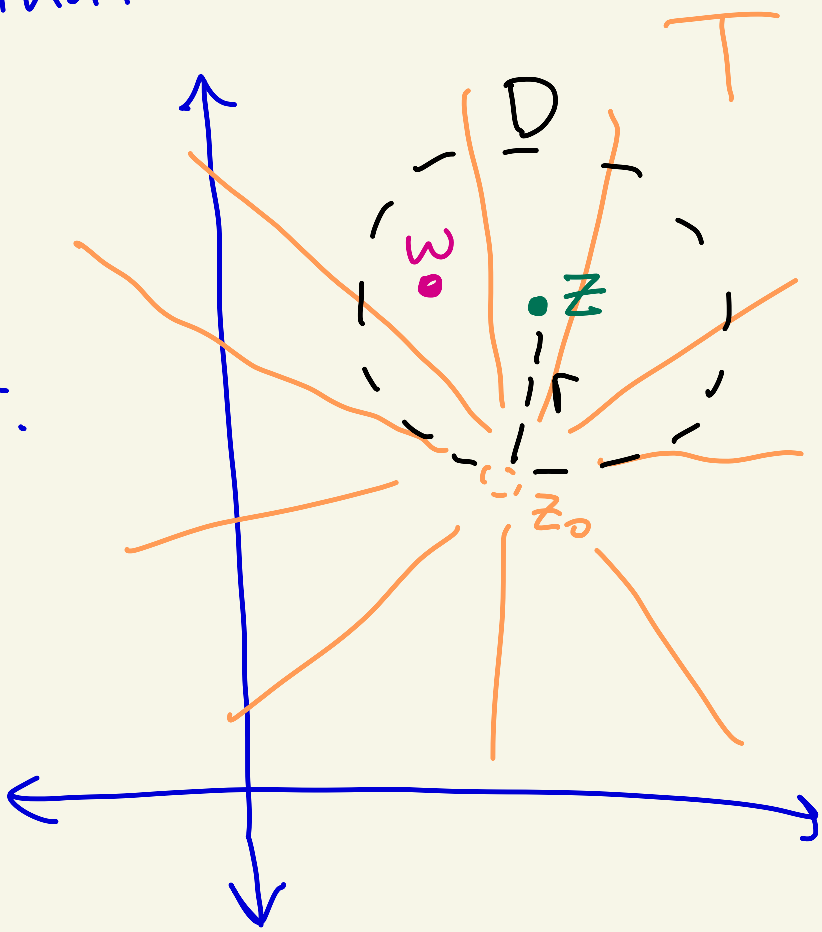
$\boxed{3(d)}$ ϕ is closed because
$\mathbb{C} - \phi = \mathbb{C}$ is open (by 3(a)).

3(e) Let $z_0 \in \mathbb{C}$.

Let $S = \{z_0\}$.

Let $T = \mathbb{C} - S = \mathbb{C} - \{z_0\}$

We want to show that T is open.



Let $z \in T$.

We want to show that z is an interior point of T.

Let $r = |z - z_0|$.

Let $D = D(z; r)$.

We want to show that $D \subseteq T$.

Let $w \in D = D(z;r) = \{w \mid |w-z| < r\}$.

Then $|w - z| < r$.

To show that $w \in T$ we need to show that $w \neq z_0$

Suppose $w = z_0$.

Then

since $w \in D$

$$r = |z - z_0| = |z - w| < r$$

So, $r < r$

Contradiction.

Thus, $w \neq z_0$.

So, $w \in T = \mathbb{C} - \{z_0\}$.

Thus, $D \subseteq T$.

So, z is an interior point of T

So, T is open.

$\boxed{3(f)}$ Let A and B be open sets in $\mathbb{C}$.

If $A \cap B = \phi$, then $A \cap B$ is open by problem 3(b).

Hence, we may assume $A \cap B \neq \phi$.

Let $z \in A \cap B$.

We show that z is an interior point of $A \cap B$.

Since $z \in A \cap B$ we know $z \in A$ and $z \in B$.

Since $z \in A$ and A is open, we have that z is an interior point of A.

So, $\exists r_1 > 0$ so that $D(z; r_1) \subseteq A$

Similarly, since $z \in B$ and B is open then $\exists r_2 > 0$ with $D(z; r_2) \subseteq B$.

Let $r = \min\{r_1, r_2\}$.

Then,

$D(z;r) \subseteq D(z;r_1) \subseteq A$ and $D(z;r) \subseteq D(z;r_2) \subseteq B$

So, $D(z;r) \subseteq A \cap B$. So, z is an interior point of $A \cap B$. ▨

$\boxed{3(g)}$ Let A and B be closed subsets of $\mathbb{C}$. Then

$\mathbb{C} - A$ is open and

$\mathbb{C} - B$ is open.

De Morgan set theory
$(A \cup B)^c = A^c \cap B^c$

Thus,

$$\mathbb{C} - (A \cup B) = (\mathbb{C} - A) \cap (\mathbb{C} - B)$$

By 3(f) we have that $\mathbb{C} - (A \cup B)$ is open. $\blacksquare$